

Asymptotically Lifshitz Brane-World Black Holes

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Abstract

We study the gravity dual of a Lifshitz field theory in the context of a RSII brane-world scenario, taking into account the effects of the extra dimension through the contribution of the electric part of the Weyl tensor. We show that although the Lifshitz space-time cannot be considered as a vacuum solution of the RSII brane-world, the asymptotically Lifshitz solution can. We then study the thermodynamical behavior of such asymptotically Lifshitz black holes. It is shown that the condition on the positivity of entropy imposes an upper bound on the critical exponent z . This maximum value of z corresponds to a positive infinite entropy as long as the temperature is kept positive. The stability and phase transition for different spatial topologies are also discussed.

1 Introduction

Quantum field theories and their holographic duals in the context of AdS/CFT have been the subject of numerous studies in recent years. The AdS/CFT correspondence provides a well established method for obtaining dual descriptions of strongly coupled conformal field theories in terms of weakly coupled gravitational theories where the gravitational background symmetries are realized as the symmetries of the dual CFT. For example, the conformal group $SO(D, 2)$ of a D dimensional CFT arises as the isometry group of AdS_{D+1} .

The AdS/CFT seems to extend to some systems which do not exhibit Lorentz invariance. Recently, attempts have been made to apply the holographic principle to study condensed matter systems near a critical point [1, 2] where there are many strongly coupled systems which have to be studied non-perturbatively. In this situation, one encounters many scale-invariant field theories which are not Lorentz invariant. In such theories, space and time are not considered in the same footing, so they can scale in different ways under dilatation

$$t \rightarrow \lambda^z t, \quad \vec{x} \rightarrow \lambda \vec{x}, \quad (1)$$

where z is the so-called dynamical critical exponent showing relative scale dimension of space and time. For $z = 1$, this scaling symmetry reduces to the familiar relativistic scale invariance

$$t \rightarrow \lambda t, \quad \vec{x} \rightarrow \lambda \vec{x}. \quad (2)$$

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The gravitational dual of theories with non-relativistic scale invariant transformation (1) has been already considered with Schrödinger symmetry [3, 4] and Lifshitz symmetry [5].

In this paper, we focus our attention on the Lifshitz theory as an anisotropic scale-invariant theory. A toy model which exhibits such anisotropic scale invariance is the Lifshitz field theory represented by

$$\mathcal{L} = \int d^2x dt \left[(\partial_t \phi)^2 - \kappa (\nabla^2 \phi)^2 \right]. \quad (3)$$

The theory describes a line of fixed points parameterized by κ and respects the scaling invariance (1) with $z = 2$.

Generally speaking, theories assuming scaling invariance as well as time and space translations, spatial rotation, parity and time reversal symmetries, admit a metric of the form [5, 6, 7]

$$ds^2 = l^2 \left(-r^{2z} dt^2 + \frac{dr^2}{r^2} + r^2 dx_i^2 \right), \quad i = 1, \dots, d \quad (4)$$

where l is a characteristic length scale and d is the number of spatial dimensions of the space-time in the subspace of constant t and r .

The Einstein-Hilbert (EH) action cannot admit a Lifshitz geometry [5, 6, 8, 9, 10]. However, there are generically two ways to construct actions which admit such geometry (4). One way is by adding some nontrivial matter terms to the EH action [5]. Alternatively, one may add higher order curvature terms to generate dynamics for parameter z to ensure that the Lifshitz geometry (4) satisfies the resulting field equations of the new action [8, 9, 10, 11]. We refer to these procedures as a renormalization of z . It is generally difficult to obtain analytic solutions to the field equations generated by these actions. Attempts have been made to find analytic black hole solutions which asymptotically admit Lifshitz geometry [12, 13, 14]. We propose an alternative to renormalize the dynamical exponent z , considering the global effects of the extra dimensions. Consequently, asymptotically Lifshitz black hole solutions can be obtained in the context of brane-world scenarios which is the main purpose of the present paper.

Another important direction in gravitational theories emerged a little more than a decade ago by the advent of what is now known as brane-world models. The primary idea behind brane-world models was introduced by Rubakov and Shaposhnikov with the suggestion that the confinement of gauge fields to a brane can be achieved by means of a potential well which is narrow along the extra dimensions and flat along the others [15]. The idea was followed by Randall and Sundrum [16] (RS) when they proposed a way to localize gravity around the brane hypersurface and hence explained the hierarchy problem of the fundamental forces. In an RS-type brane world theory, localization of gravity around the brane is achieved by assuming an AdS_5 geometry for the bulk space-time. Among the two RS models for which two or one brane are considered, the latter became more favorable and its gravitational consequences have largely been investigated. The first attempt to write the equations of motion governing gravity on such branes was initiated by Shiromizu, Maeda and Sasaki (SMS) [17]. They assumed general bulk and brane geometries and obtained equations of motion of the matter field on the brane by use of the Gauss-Codazzi equations and Israel junction conditions [18]. The resulting field equations include the global bulk structure through the electric part of the Weyl tensor. The main feature of these equations is the introduction of a new tensor which is quadratic in the energy-momentum tensor. As already mentioned, in this paper we concentrate our attention on black hole solutions resulting from the field equations on the brane, obtained by the application of the SMS procedure. The difference between such a brane equation with the standard Einstein equation in vacuum is the appearance

of an additional bulk effect through the electric part of the Weyl tensor. This tensor can be decomposed in $4D$ to three tensors by a given time-like vector field known as dark radiation, dark energy flux and dark pressure [19].

The brane generalization of the vacuum Einstein field equations (4) does not admit the Lifshitz geometry. Roughly speaking, this arises from the dependence of matter components to the couplings of the theory. This implies that a Lifshitz space-time cannot be admitted as a solution by the field equations on the brane in the absence of an energy-momentum tensor. The calculations leading to this result are presented in the Appendix. However, as we will show below, the brane field equations can admit asymptotically Lifshitz geometry. This is done with the aid of the electric part of the Weyl tensor which ensures the renormalizability of the dynamical exponent z . It is worth mentioning that attempts have been made to show that the Lifshitz space-time can be a solution of a brane-world model in the presence of an anisotropic perfect fluid energy-momentum tensor on a thick brane [20, 21].

The paper is organized as follows: In the next two sections we obtain the asymptotically Lifshitz solution to the brane geometry and show that the value of the dynamical exponent must be bounded from below to represent a black hole solution. We interpret the constants of the solution in terms of the mass and charge of a black hole using the fact that in the case $z = 1$ the solutions reduce to that of the familiar black holes. We then discuss the position of the horizons in some interesting cases. In section 4 we study the thermodynamical properties of such black holes. As we will see, the entropy restricts the value of the dynamical exponent from above. This result shows that the asymptotically Lifshitz brane black holes only admit a narrow range for z . The stability of the black hole and the possibility of phase transition in some interesting cases of z will also be discussed. The reasons for the lack of a vacuum Lifshitz solution in the present framework are discussed in the Appendix. Conclusions are drawn in the last section.

2 The setup

Brane-world scenarios begin with a $5D$ action of the form

$$S = \int d^5x \sqrt{-{}^5g} {}^5R, \quad (5)$$

which using the Gauss-Codazzi equations and Israel junction conditions can be written in the form of a generalized Einstein field equation [17]

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + \kappa_4^2 \tau_{\mu\nu} + \kappa_5^4 \pi_{\mu\nu} - \mathcal{E}_{\mu\nu}, \quad (6)$$

where

$$\pi_{\mu\nu} = -\frac{1}{4} \tau_{\mu\alpha} \tau^\alpha_\nu + \frac{1}{12} \tau \tau_{\mu\nu} + \frac{1}{8} g_{\mu\nu} \tau_{\alpha\beta} \tau^{\alpha\beta} - \frac{1}{24} g_{\mu\nu} \tau^2, \quad (7)$$

and $\tau_{\mu\nu}$ is the energy-momentum tensor of the matter fields residing on the brane and $\mathcal{E}_{\mu\nu}$ is the electric part of the Weyl tensor.

Let us use the following irreducible decomposition for the electric part of the Weyl tensor [19]

$$\mathcal{E}_{\mu\nu} = -\kappa^4 \left[U \left(u_\mu u_\nu + \frac{1}{3} h_{\mu\nu} \right) + \mathcal{P}_{\mu\nu} + 2\mathcal{Q}_{(\mu} u_{\nu)} \right], \quad (8)$$

where $\kappa = \frac{\kappa_5}{\kappa_4}$, and the spatial metric $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ is orthogonal to the chosen time-like vector field u^μ . In this decomposition, we introduce three terms, known as dark radiation

$$U = -\kappa^{-4} \mathcal{E}_{\mu\nu} u^\mu u^\nu, \quad (9)$$

dark energy flux

$$\mathcal{Q}_\mu = \kappa^{-4} h_\mu^\alpha \mathcal{E}_{\alpha\beta} u^\beta, \quad (10)$$

and dark pressure

$$\mathcal{P}_{\mu\nu} = -\kappa^{-4} \left[h_{(\mu}^\alpha h_{\nu)}^\beta - \frac{1}{3} h_{\mu\nu} h^{\alpha\beta} \right] \mathcal{E}_{\alpha\beta}. \quad (11)$$

In the case of a static and isotropic space-time we have $\mathcal{Q}_\mu = 0$, and

$$\mathcal{P}_{\mu\nu} = P(r) \left[r_\mu r_\nu - \frac{1}{3} h_{\mu\nu} \right], \quad (12)$$

where r_μ is the unit radial vector [22].

3 Asymptotically Lifshitz Black hole Solution

To obtain the black hole solutions which are asymptotic to the Lifshitz space-time for large r , we assume that the brane metric is given by

$$ds^2 = -f(r) \left(\frac{r}{l} \right)^{2z} dt^2 + \frac{1}{f(r)} \left(\frac{l}{r} \right)^2 dr^2 + r^2 d\Omega^2, \quad (13)$$

where $d\Omega^2$ is the $2D$ portion of the metric which depends on the curvature index of the space-time

$$d\Omega^2 = d\theta^2 + \begin{cases} \sin^2 \theta d\phi^2 & k = +1 \\ \theta^2 d\phi^2 & k = 0 \\ \sinh^2 \theta d\phi^2 & k = -1 \end{cases}, \quad (14)$$

and, as was mentioned in the Introduction, z shows the relative scaling of the space and time. This form of the metric guarantees that the brane geometry approaches the Lifshitz geometry for $d = 2$, equation (4), up to the re-scaling $r \rightarrow r/l$ and $t \rightarrow t/l$ if $f(r) \rightarrow 1$ as $r \rightarrow \infty$.

We also assume that the brane energy-momentum tensor is zero, so that the brane equations of motion reduce to the form

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + \mathcal{E}_{\mu\nu} = 0. \quad (15)$$

Assuming $u^\mu = \delta_0^\mu$ and $r^\mu = \delta_1^\mu$, the electric part of the Weyl tensor can be computed from metric (13) as

$$\mathcal{E}_0^0 = \kappa^4 U(r), \quad \mathcal{E}_1^1 = -\frac{1}{3} \kappa^4 [U(r) + 2P(r)], \quad \mathcal{E}_2^2 = \mathcal{E}_3^3 = -\frac{1}{3} \kappa^4 [U(r) - P(r)]. \quad (16)$$

Using equations (13), (15) and (16) one can write the Einstein equations

$$\begin{aligned} \frac{r f' + 3f}{l^2} - \frac{k}{r^2} + \kappa^4 U(r) + \Lambda &= 0, \\ \frac{r f' + (1+2z)f}{l^2} - \frac{k}{r^2} - \frac{1}{3} \kappa^4 [U(r) + 2P(r)] + \Lambda &= 0, \\ \frac{3}{2} \frac{1+z}{l^2} r f' + \frac{z(1+z)}{l^2} f + \frac{1}{2l^2} r^2 f'' - \frac{1}{3} \kappa^4 [U(r) - P(r)] + \Lambda &= 0. \end{aligned} \quad (17)$$

These equations can be solved for $f(r)$ with the result

$$f(r) = -\frac{2\Lambda l^2}{z^2 + 2z + 3} + \frac{k}{1 - z + z^2} \left(\frac{l}{r}\right)^2 + C_2 \left(\frac{l}{r}\right)^{B_2} + C_3 \left(\frac{l}{r}\right)^{B_3}, \quad (18)$$

where we have defined

$$\begin{aligned} B_2 &\equiv 2 + \frac{3z}{2} + \sqrt{\frac{z^2}{4} + 2z - 2}, \\ B_3 &\equiv 2 + \frac{3z}{2} - \sqrt{\frac{z^2}{4} + 2z - 2}, \end{aligned} \quad (19)$$

with both C_2 and C_3 being constants. To have an asymptotic Lifshitz black hole, the leading term of the function $f(r)$ must be equal to unity and the constants B_2 and B_3 must be well defined and positive. This means that in a brane-world scenario the dynamical exponent is bounded from below,

$$z \geq -4 + 2\sqrt{6}, \quad (20)$$

and the cosmological constant could be read off as a function of the dynamical exponent

$$\Lambda = -\frac{1}{2} \frac{z^2 + 2z + 3}{l^2}. \quad (21)$$

Now, the calculations would become much simpler if one writes $f(r)$ in the following form

$$f(r) = \sum_{\mu=0}^3 C_\mu \left(\frac{l}{r}\right)^{B_\mu}, \quad (22)$$

and defines

$$B_0 = 0, \quad B_1 = 2, \quad C_0 = 1, \quad C_1 = \frac{k}{1 - z + z^2}. \quad (23)$$

It is now possible to calculate dark radiation and pressure from equation (17) as

$$\kappa^4 U(r) = -\Lambda + \frac{k}{r^2} - \sum_{\mu=0}^3 \frac{C_\mu(3 - B_\mu)}{l^2} \left(\frac{l}{r}\right)^{B_\mu}, \quad (24)$$

$$\kappa^4 P(r) = 2\Lambda - \frac{2k}{r^2} + \sum_{\mu=0}^3 \frac{C_\mu(3 - 2B_\mu + 3z)}{l^2} \left(\frac{l}{r}\right)^{B_\mu}. \quad (25)$$

Since the integration constants C_2 and C_3 are independent of z they can be obtained by considering the behavior of the solution at $z = 1$. This is the case where the relativistic scaling invariance is realized and the resulting solution can be compared with the known black hole solutions. In this case, the asymptotic Lifshitz solution of the brane reduces to the form

$$ds^2 = - \left(k + \frac{r^2}{l^2} + \frac{C_3 l}{r} + \frac{C_2 l^2}{r^2} \right) dt^2 + \frac{1}{\left(k + \frac{r^2}{l^2} + \frac{C_3 l}{r} + \frac{C_2 l^2}{r^2} \right)} dr^2 + r^2 d\Omega^2, \quad (26)$$

where $d\Omega^2$ is given by equation (14) and $\Lambda = -\frac{3}{l^2}$. Comparing with the standard topological Reissner-Nordström-AdS black hole solution

$$ds^2 = - \left(k - \frac{\Lambda}{3} r^2 - \frac{2M}{r} + \frac{Q^2}{r^2} \right) dt^2 + \frac{1}{\left(k - \frac{\Lambda}{3} r^2 - \frac{2M}{r} + \frac{Q^2}{r^2} \right)} dr^2 + r^2 d\Omega^2, \quad (27)$$

one can read off the constants as

$$C_2 = \frac{Q^2}{l^2} \equiv q^2, \quad C_3 = -\frac{2M}{l} \equiv -2m. \quad (28)$$

From now on, m and q are referred to as the mass and charge parameters of the general asymptotic Lifshitz black hole (26).

One may obtain the position of the horizon r_H of the black hole through imposing the condition $f(r_H) = 0$, which for the solution (22) leads to

$$u^{2+\frac{3z}{2}} + C_1 u^{\frac{3z}{2}} + q^2 u^{-b} - 2m u^b = 0, \quad (29)$$

where we have defined

$$b = \sqrt{\frac{z^2}{4} + 2z - 2}, \quad u = \frac{r_H}{l}. \quad (30)$$

Due to the form of the metric components (22) one cannot compute the position of the horizon analytically. It is therefore appropriate to study the position of the horizon in some special cases. In what follows, we shall concentrate on the solution of equation (29) for values of the critical exponent z , given by $z = -4 + 2\sqrt{6}$ which is its minimum value, $z = 1$, and also $z = 2$ that is near the maximum value for z , obtained from thermodynamic considerations in the next section, and therefore expected to have a similar behavior.

3.1 $z = -4 + 2\sqrt{6}$

For this value of z which is the minimum possible value for the dynamical exponent, equation (29) reduces to

$$u^2 + \frac{1}{9}(5 + 2\sqrt{6})k + au^{6-3\sqrt{6}} = 0, \quad (31)$$

where we have defined $a = q^2 - 2m$. In the case $k = 0$ one has an exact solution for the position of the horizon, namely

$$u = (-a)^{\frac{1}{-4+3\sqrt{6}}}, \quad (32)$$

which implies that in order for the black hole to have a horizon, the quantity a must be negative. This situation holds also in the case $k = 1$. However, the case $k = -1$ has an exception for some small values of a . In figure (1), the quantity a has been plotted in terms of the position of the horizon for two cases $k = \pm 1$. As is shown in this figure, for $k = 1$ all black holes have only one horizon which corresponds to a negative value of a ; the greater the ratio of the black hole mass to its charge, the greater the black hole horizon. In the case $k = -1$, however, positive values for a are possible and lead to two horizons until the quantity a reaches its maximum value at $a = 0.4$. In this situation we have an extremal black hole with the horizon radius $r = 0.665l$. In general, the position of the radius of the extremal black hole is given by the following equation

$$B_3 + C_1(B_3 - 2)u^{-2} + C_2(B_3 - B_2)u^{-B_2} = 0. \quad (33)$$

The negative values of a lead to black holes with only one horizon, which is similar to the case $k = 1$.

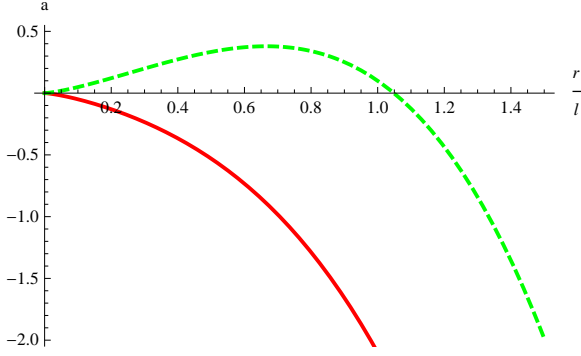


Figure 1: Plot of a as a function of $\frac{r_H}{l}$ for $z = -4 + 2\sqrt{6}$. Solid (Dashed) lines correspond to $k = 1$ ($k = -1$) respectively.

3.2 $z = 1$

This case is interesting because we recover the relativistic scaling symmetry. In addition, this is the case where we have an exact solution. Here, equation (29) reduces to

$$u^4 + ku^2 - 2mu + q^2 = 0. \quad (34)$$

This equation has an exact solution corresponding to the position of the horizons

$$u = -\frac{1}{2}\sqrt{\xi} + \frac{1}{2}\sqrt{-2k - \xi - \frac{4m}{\sqrt{\xi}}}, \quad (35)$$

$$u = +\frac{1}{2}\sqrt{\xi} \pm \frac{1}{2}\sqrt{-2k - \xi + \frac{4m}{\sqrt{\xi}}}, \quad (36)$$

where we have defined

$$\xi = -\frac{2k}{3} + \frac{\sqrt[3]{2}(k^2 + 12q^2)}{3\eta} + \frac{\eta}{3\sqrt[3]{2}}, \quad (37)$$

$$\eta = \sqrt[3]{\gamma + \sqrt{-4(k^2 + 12q^2)^3 + \gamma^2}}, \quad (38)$$

$$\gamma = 2k^3 + 108m^2 - 72kq^2. \quad (39)$$

The existence of the horizon lies in the positivity of the quantity ξ . This condition leads to the following range for the mass and charge parameters of the black hole

$$q^2 < f(m, k), \quad (40)$$

where

$$f(m, k) = \frac{-108km^2 + (k^2 + \xi_1)^2}{12\xi_1}, \quad (41)$$

with

$$\xi_1 = \left(1458m^4 - 270km^2 - k^2 + \sqrt{108m^2(2k + 27m^2)^3} \right)^{1/3}, \quad (42)$$

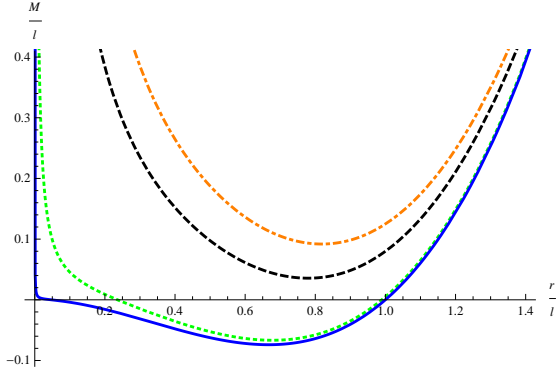


Figure 2: The mass parameter for $k = -1$ and $z = 1$ as a function of the horizon radius. Solid, dotted, dashed and dashed-dotted lines correspond to $q = 0.01, 0.1, 0.4$ and 0.5 respectively.

provided

$$m \geq \frac{1}{6\sqrt{3}}, \quad k = 1, \quad (43)$$

$$m \geq \frac{2}{3\sqrt{6}}, \quad k = -1. \quad (44)$$

For $k = 1$ and $k = 0$, there are black holes with only one horizon, if

$$\xi \leq \frac{4m}{\sqrt{\xi}} - 2k < 2\xi, \quad (45)$$

and with two horizons, if

$$\frac{4m}{\sqrt{\xi}} \geq 2(\xi + k). \quad (46)$$

The case $k = -1$ is more interesting. For the existence of a horizon one must have

$$\frac{4m}{\sqrt{\xi}} < -\xi + 2, \quad (47)$$

which implies $\xi < 2$. In addition, if we have $\xi < 1$ and $\frac{4m}{\sqrt{\xi}} < -2\xi + 2$, we will obtain only one horizon. However, for $1 < \xi < 2$ and $\frac{4m}{\sqrt{\xi}} < 2\xi - 2$ we obtain two horizons. Figure (2) shows the mass parameter as a function of the horizon radius for $k = -1$. As we can see from the figure, for small values of q the black hole acquires two horizons which move farther apart for larger m . As q reaches some critical value, a black hole starts to form as an extremal black hole and, for larger values of m the black hole acquires two horizons as in the case of Reissner-Nordström black holes in the standard GR. One of the ingredients of these solutions is that the position of the outer horizon becomes independent of the value of parameter q . This property is common to all values of k .

3.3 $z = 2$

Finally, we discuss the case $z = 2$. As we shall see in the next section, $z = \frac{11}{5}$ is the maximum value for the dynamical exponent, implying the positivity of the entropy of the corresponding

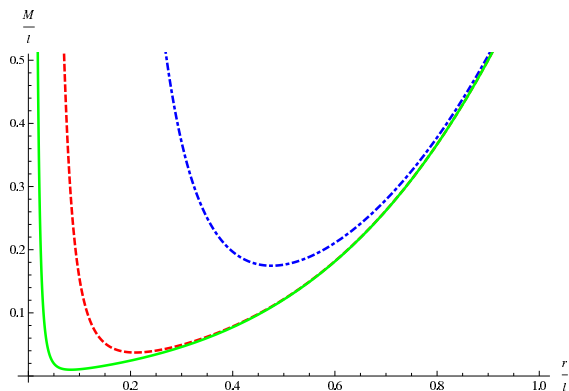


Figure 3: The mass parameter for $z = 2$, $k = 1$ and $q = 0.001$ (solid), $q = 0.01$ (dashed), $q = 0.1$ (dot-dashed).

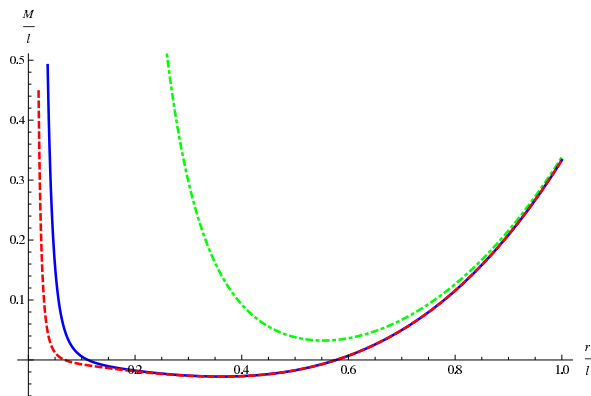


Figure 4: The mass parameter for $z = 2$, $k = -1$ and $q = 0.001$ (dashed), $q = 0.01$ (solid), $q = 0.1$ (dot-dashed)

thermodynamical system. For maximum value of z , the entropy becomes infinite, corresponding to highly stable black holes. The value $z = 2$ is very close to the maximum value and exhibits the same behavior. In this case equation (29) reduces to

$$u^5 + \frac{1}{3}ku^3 + q^2u^{-\sqrt{3}} - 2mu^{\sqrt{3}} = 0. \quad (48)$$

In figures (3) and (4) we have plotted the mass parameter as a function of the horizon radius for three values of q for $k = \pm 1$. The case $k = 0$ is quite similar to $k = 1$. As we can see from the figures, for larger values of the parameter q , the distance between the inner and outer horizons, indicated by the intersections of a horizontal line of constant m with a curve, decreases. As for $z = 1$, this case also has the property that the mass parameter for large mass black holes is independent of the q parameter. The case $k = -1$ is however more interesting. For the small values of the parameter q , we do not have an extremal black hole. The larger values of q imply that extremal black holes do exist.

4 Thermodynamical properties

In this section we study the thermodynamical behavior of the asymptotically Lifshitz black holes (13). After calculating the corresponding thermodynamical quantities for such black holes we

discuss their stability.

The gravitational mass of a black hole, M , is obtained by the equation $f(r_H) = 0$, governing the position of the horizon. In the case of black hole (13) we find

$$M = \frac{l}{2} u^{B_3} \sum_{\mu=0}^2 C_\mu u^{-B_\mu}, \quad (49)$$

where u is associated with the outer most horizon.

The Hawking temperature associated with a black hole is obtained by requiring the absence of a conical singularity at the black hole horizon in the Euclidean sector of the asymptotically Lifshitz black hole. One then has

$$T = \frac{\kappa}{2\pi}, \quad (50)$$

where the surface gravity κ on the horizon admits the following useful representation

$$\kappa^2 = -\frac{1}{2} \xi_{\alpha;\beta} \xi^{\alpha;\beta}, \quad (51)$$

where ξ is a killing vector normal to the surface of the horizon. As a consequence, the surface gravity of a black hole could be written in terms of the temporal component of the metric as

$$\kappa = -\frac{1}{2} \frac{dg_{tt}}{dr} \Big|_{r=r_H}. \quad (52)$$

The Hawking temperature of the asymptotically Lifshitz black hole (13) is now given by

$$T = \frac{1}{4\pi l} u^{2z-1} \sum_{\mu=0}^2 C_\mu (B_3 - B_\mu) u^{-B_\mu}. \quad (53)$$

One should note that for asymptotically flat space-times, the normalization of the killing vector field, ξ , in (51) is imposed at the spatial infinity. However, it has been shown by Brown *et al.* that the choice of the normalization elsewhere results in the appearance of an additional redshift factor, the so-called Tolman redshift factor, for a given temperature in a stationary gravitational field [23]. For non-asymptotically flat space-times, one should impose the normalization at a finite space-like boundary surface. Nevertheless, one can assume that observations take place at the boundary surface where the redshift factor equals unity. So, in what follows we set the Tolman redshift factor equal to “one” without loss of generality.

Another important thermodynamical quantity is the entropy of black holes. In Einstein gravity the entropy of a black hole is obtained by the so-called area formula. This means that the entropy is equal to one-quarter of the horizon area. It has also been shown that when higher curvature terms are present, this statement is no longer true [24]. However, Wald [25] has shown that in any case the entropy of a black hole is a function of the horizon geometry. Since the black hole as a thermodynamical system must obey the first law of thermodynamics, $dM = TdS$, there is an alternative equivalent way to define the entropy of black hole via

$$S = \int_0^{r_H} \frac{1}{T} \frac{\partial M}{\partial r} dr. \quad (54)$$

Using (49) and (53) we find

$$S = \frac{2\pi l^2}{B_3 - 2z + 1} u^{B_3 - 2z + 1}. \quad (55)$$

The positivity condition on the entropy implies an upper bound on the dynamical critical exponent z , that is

$$z < \frac{11}{5}. \quad (56)$$

From equation (55) we see that $z = \frac{11}{5}$ corresponds to an infinitely large entropy. It then implies that the dynamical exponent z must lie near this value by the maximum entropy principle. This behavior however resembles some condensed matter systems such as Rokhsar-Kivelson dimer model [26] and the strongly correlated electron systems [27], which is well described by the action (3) at $z = 2$. From the view point of duality, our result is expectable since our gravitational model becomes strongly stable near the point for which its dual theory becomes stable and physical. It is worth mentioning that the limit (56) is obtained in the context of asymptotically Lifshitz space-times which may induce a perturbation term on the dual field theory (3). This perturbation term can cause the critical exponent admitting values up to $z = \frac{11}{5}$, supporting our gravitational model. As a result, our model restricts the physically reasonable solution to those which admit a dynamical exponent z in the range

$$-4 + 2\sqrt{6} \leq z < \frac{11}{5}. \quad (57)$$

However, if we do not take the condition (56) *ab initio*, the entropy will contain negative values. This happens, for example, in the case of the Gauss-Bonnet black holes for negative spatial curvature. It has been emphasized that negative values for the entropy is related to the lower limit of the integral (54). This problem could be resolved by considering a minimum radius r_{min} corresponding to the zero gravitational mass as the lower limit of the integral [24]. In the case of the black hole (13), the entropy is independent of the spatial curvature so that it can be negative for all topologies. In our case we remove negative values for the entropy by restricting the dynamical exponent z .

In order to study the thermodynamical stability of black holes, one should calculate the heat capacity. The local thermal stability of a black hole is determined by the sign of the heat capacity. If the heat capacity is positive, the black hole is locally stable to thermal fluctuations. Negative values for the heat capacity result in unstable black holes. We define the heat capacity for a constant charge black hole as

$$C_Q \equiv \left(\frac{\partial M}{\partial T} \right)_Q = \left(\frac{\partial M}{\partial r_H} \right)_Q \left(\frac{\partial r_H}{\partial T} \right)_Q. \quad (58)$$

For the black hole (13), the heat capacity can be written as

$$C_Q = 2\pi l^2 u^{B_3 - 2z + 1} \frac{\sum_{\mu=0}^2 C_\mu (B_3 - B_\mu) u^{-B_\mu}}{\sum_{\mu=0}^2 C_\mu (B_3 - B_\mu) (2z - 1 - B_\mu) u^{-B_\mu}}. \quad (59)$$

This expression may be converted to a more familiar thermodynamical relation

$$\frac{1}{C_Q} = \frac{1}{2\pi l^2} u^{2z - B_3} \left[\frac{\partial}{\partial u} \ln(4\pi T) \right]. \quad (60)$$

The free energy of a black hole is defined as $F = M - TS$. Using equations (49), (53) and (54) which are the expressions for gravitational mass, temperature and entropy respectively, one obtains the free energy

$$F = \frac{l}{2} u^{B_3} \sum_{\mu=0}^2 C_\mu \left[\frac{B_\mu - 2z + 1}{B_3 - 2z + 1} \right] u^{-B_\mu}. \quad (61)$$

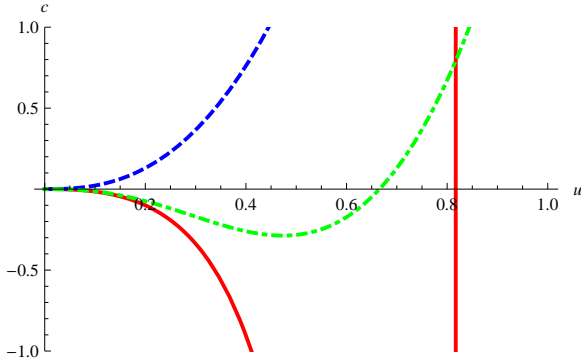


Figure 5: Normalized heat capacity c_q for $z = -4 + 2\sqrt{6}$ versus the event horizon of the black hole. The solid, dashed and dot-dashed curves correspond to $k = 1$, $k = 0$ and $k = -1$ respectively.

The value of the free energy determines the strength of the stability of a black hole; a more stable black hole corresponds to less free energy.

As we will see in the following discussion, in most cases, there are some regions for which the heat capacity is positive and consequently the black hole is stable in those regions. Since there is no way to determine the sign of the heat capacity analytically in the case of the black hole (13), we focus our attention in studying the behavior of the heat capacity on the following three interesting cases of z , namely, for minimum possible value of z , i.e. $z = -4 + 2\sqrt{6}$, $z = 1$ and $z = 2$ which is close to the maximum value $z = \frac{11}{5}$ and reflects the behavior of black holes with maximum entropy.

4.1 $z = -4 + 2\sqrt{6}$

In the case of minimum z , the heat capacity has the form

$$c_q = \frac{C_Q}{l^2} = 2\pi \frac{[168u^2 - 69u^2\sqrt{6} + (2 - \sqrt{6})k] u^{5-\sqrt{6}}}{-3168u^2 + 1293u^2\sqrt{6} - (46 - 19\sqrt{6})k}. \quad (62)$$

It is interesting to note that the heat capacity is charge-independent. Figure (5) shows the behavior of the normalized heat capacity c_q in terms of the horizon radius for different values of spatial curvature. For $k = 0$, the heat capacity is positive for all black hole horizon radii. For the case of topological black holes ($k = -1$), those with a horizon larger than $u \sim 0.7$ are stable, and those with horizons smaller than this value are always unstable. However, the most interesting case is the one with $k = +1$. In this situation, the heat capacity exhibits a singularity at the point $u = 0.8168$. This point corresponds to a phase transition for the black hole which will be discussed below. Black holes with a horizon larger than this value are thermally stable and those with smaller values are unstable. The strength of the stability of a black hole is determined by the corresponding free energy which is given by

$$f = \frac{F}{l} = \frac{1}{2}q^2 + \frac{1}{38} (47 + 17\sqrt{6}) \left(\frac{1}{9}ku^{-2} + \frac{1}{25} (111 - 46\sqrt{6}) \right) u^{-4+3\sqrt{6}}. \quad (63)$$

Figure (6) shows the behavior of the normalized free energy f for different horizon radii. The results show that while the behavior of large radius black holes is not so sensitive to the charge of the black hole, the behavior of a small radius black hole depends on the value of the electric charge. One can see from the figure that the large radius black holes are more stable.

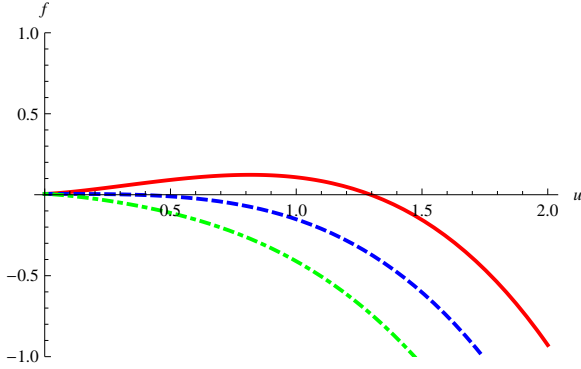


Figure 6: Normalized free energy f for $z = -4 + 2\sqrt{6}$ and $q = 0.1$ versus the event horizon of the black hole. The solid, dashed and dot-dashed curves correspond to $k = 1$, $k = 0$ and $k = -1$ respectively.

Moreover, there is another interesting result. The singularity in the heat capacity corresponds to a phase transition provided that the temperature and free energy possess a local minimum and a local maximum at the specified points respectively. In order to make sure that this singular point corresponds to a phase transition, we ought to check this statement. Figure (7) shows the derivative of temperature as a function of the horizon radius, derivative of the free energy, again as a function of the horizon radius and the heat capacity for a black hole with minimum z and $k = 1$. The crossing of the minimum temperature, maximum free energy and singularity of the heat capacity at a point implies that there is one physically reasonable phase transition at $u = 0.8168$. We note that in this case, the black hole experiences a phase transition when its event horizon becomes of the order of Lifshitz characteristic length scale.

At this point it is appropriate to remark on the conditions for the occurrence of any physical phase transitions in Lifshitz theories. It has been shown in [28] that in order to have a physical phase transition in Lifshitz theories which is described by [5], one must have another length scale in addition to the Lifshitz length scale. This additional length scale was achieved by introducing new Maxwell fields into the theory. Nonetheless, the brane world scenario discussed here has an intrinsic length scale introduced by the ratio of the brane to the bulk Planck scales. In this context therefore, it is possible to have a physical phase transition.

4.2 $z = 1$

Let us now study the case $z = 1$ where we expect the black hole to reduce to that of a topological Reissner-Nordström-AdS type. In this case the normalized temperature $\Theta = Tl$, free energy and heat capacity reduce to the following simple forms

$$\begin{aligned}\Theta &= \frac{1}{4\pi}u(3 + ku^{-2} - q^2u^{-4}), \\ f &= \frac{1}{4}u^3(-1 + ku^{-2} + 3q^2u^{-4}), \\ c_q &= 2\pi \frac{u^2(3 + ku^{-2} - q^2u^{-4})}{3 - ku^{-2} + 3q^2u^{-4}}.\end{aligned}\tag{64}$$

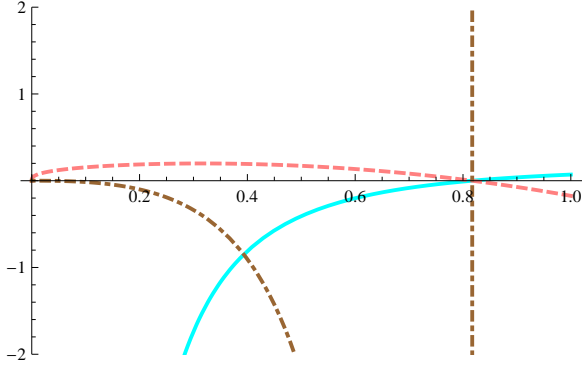


Figure 7: Derivative of temperature, of free energy and the heat capacity for $z = -4 + 2\sqrt{6}$ and $k = 1$ versus the event horizon of the black hole. The solid, dashed and dot-dashed curves correspond to $\frac{d\Theta}{du}$, $\frac{df}{du}$ and c_q respectively. The point where all the curves cross exhibits a phase transition.

The heat capacity has only one acceptable root as follows

$$u_0 = \sqrt{\frac{-k + \sqrt{k^2 + 12q^2}}{6}}. \quad (65)$$

For cases $k = 0$ and $k = -1$, black holes with a horizon larger than u_0 are stable. For values lower than this, the heat capacity is negative and the black hole becomes unstable. For the case $k = 1$, the situation becomes complicated. The heat capacity becomes singular for

$$u_1 = \sqrt{\frac{k + \sqrt{k^2 - 36q^2}}{6}}, \quad (66)$$

and

$$u_2 = \sqrt{\frac{k - \sqrt{k^2 - 36q^2}}{6}}. \quad (67)$$

The heat capacity does not possess a singular point when $k = -1$ and $k = 0$. However, for $k = 1$ there are two singularities, constraining the charge of the black hole to $q \leq \frac{1}{6}$. In this case, black holes with a horizon larger than u_1 are stable. Also, there is a region between the horizon radii u_0 and u_2 where the heat capacity is positive, presenting an opportunity to find stable black holes there. Figure (8) shows the normalized heat capacity in the case $z = 1$ for three different choices of the spatial curvature. Figure (9) shows the behavior of the normalized free energy as a function of the horizon radius. As in the case of minimum z , the larger radius black hole corresponds to a more stable one.

Figure (8) also shows that the heat capacity has two singularities for $k = 1$. As was mentioned above, a singularity in the heat capacity together with a local minimum for temperature and local maximum for free energy implies a physical phase transition. Figure (10) shows the behavior of derivative of the normalized temperature, derivative of the normalized free energy and the normalized heat capacity. The figure implies that only the second singularity corresponds to a physical phase transition.

For the values of q greater than the critical value $q = \frac{1}{6}$ which controls the occurrence of the phase transition, the behavior of black holes are very similar irrespective of their spatial topology. Such black holes are stable for the horizon radius greater than u_0 .

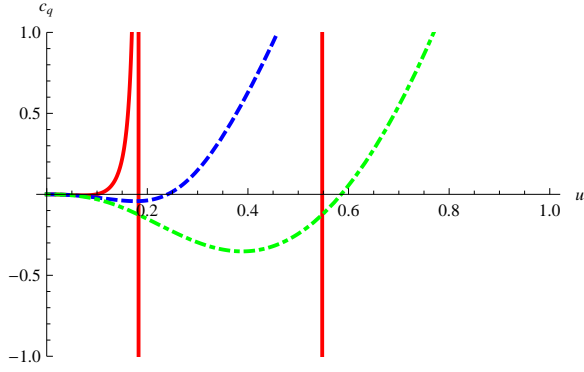


Figure 8: Normalized heat capacity c_q for $z = 1$ and $q = 0.1$ versus the event horizon of the black hole. The solid, dashed and dot-dashed curves correspond to $k = 1$, $k = 0$ and $k = -1$ respectively.

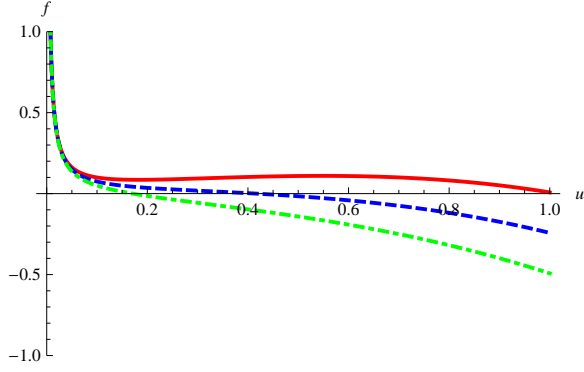


Figure 9: Normalized free energy f for $z = 1$ and $q = 0.1$ versus the event horizon of the black hole. The solid, dashed and dot-dashed curves correspond to $k = 1$, $k = 0$ and $k = -1$ respectively.

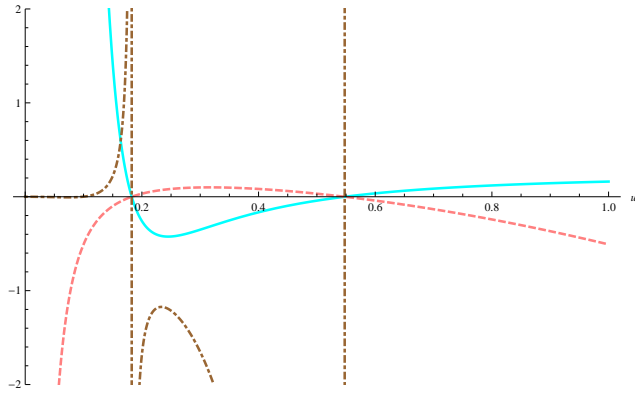


Figure 10: Derivative of the normalized temperature and of the normalized free energy and normalized heat capacity for $z = 1$ and $k = 1$ versus the event horizon of the black hole. The solid, dashed and dot-dashed curves correspond to $\frac{d\Theta}{du}$, $\frac{df}{du}$ and c_q respectively.

4.3 $z = 2$

Finally, we consider the case $z = 2$ which mimics the behavior of black holes with maximum entropy. In this case the normalized temperature, the free energy and the heat capacity reduce to

$$\begin{aligned}\Theta &= -\frac{u^3}{12\pi} \left(3\sqrt{3} - 15 - \frac{k}{u^2} (3 - \sqrt{3}) + \frac{6q^2\sqrt{3}}{u^{3+\sqrt{3}}} \right), \\ f &= \frac{9u^{5-\sqrt{3}} + u^{3-\sqrt{3}}k - 3q^2(2 + \sqrt{3})u^{-2\sqrt{3}}}{6(-2 + \sqrt{3})}, \\ c_q &= \frac{2\pi}{u\sqrt{3}} \frac{(3\sqrt{3} - 15)u^2 - k(3 - \sqrt{3}) + 6\sqrt{3}q^2u^{-3-\sqrt{3}}}{9\sqrt{3} - 45 - ku^{-2}(3 - \sqrt{3}) - 6q^2(2\sqrt{3} + 3)}.\end{aligned}\tag{68}$$

Here, the thermodynamical behavior of the black hole differs considerably depending on the charge of the black hole. Black holes with a small amount of charge ($q \sim 0.001$) show a phase transition, unlike black holes with a large amount of charge which do not.

We first consider the case $q = 0.1$. The general behavior of black holes with a large amount of charge are very similar to this case. Figure (11) shows the behavior of the heat capacity for different spatial curvatures in this case. One can see that for all horizon topologies with $k = \pm 1, 0$, black holes are stable beyond $u \sim 0.5$. As mentioned above, this value for the charge of a black hole does not imply a phase transition which can be inferred from the smoothness of the heat capacity function. Figure (12) shows the normalized free energy for all spatial topologies. One can see from the figure that a larger black hole is more stable.

As was mentioned before, the most interesting case is when the normalized charge is less than $q \sim 0.001$. Among such black holes the most interesting ones are topological black holes ($k = -1$) where a phase transition could take place in principle. For such black holes, there are two singularities in heat capacity. However the regions between these two singular points have negative temperature. The issue of a negative temperature for black holes has been investigated, for example, in [29, 30] where the physical meaning of the phase transition for such black holes is also discussed. The behavior of this type of a black hole is shown in figure (13), for the normalized heat capacity, and figure (14), for the normalized free energy. In cases $k = 0$ and $k = 1$, black holes with a horizon larger than the approximate value $r_H = 0.1l$ are stable so that the larger they get the more stable they become. For the case $k = -1$, black holes with a horizon radius in the outer region are more stable.

The maximum possible value for z corresponds to the infinitely positive entropy and this indicates that the critical exponent must eventually go close to the maximum value because of the maximum entropy principle. In this case the free energy is minus infinity everywhere as long as the temperature is positive. This confirms the last statement which implies that black holes are stable for positive temperatures, independent of the horizon radii.

5 Conclusions

Lifshitz black holes have attracted a considerable amount of attention in the past few years. However, the Lifshitz space-time is inconsistent with standard general relativity. Attempts have been made to generalize GR in such a way as to admit Lifshitz space-times. In this paper we have introduced another generalization, i.e. higher dimensions, to deal with this problem. Alas, the Lifshitz space-time is not the solution of the vacuum brane field equations. We have shown that

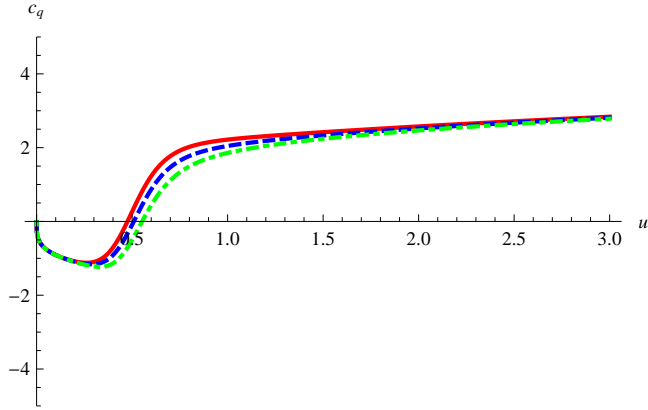


Figure 11: Normalized heat capacity c_q for $z = 2$ and $q = 0.1$ versus the event horizon of the black hole. The solid, dashed and dot-dashed curves correspond to $k = 1$, $k = 0$ and $k = -1$ respectively.

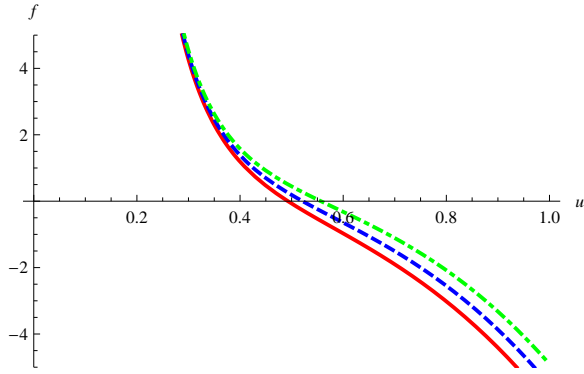


Figure 12: Normalized free energy f for $z = 2$ and $q = 0.1$ versus the event horizon of a black hole. The solid, dashed and dot-dashed curves correspond to $k = 1$, $k = 0$ and $k = -1$ respectively.

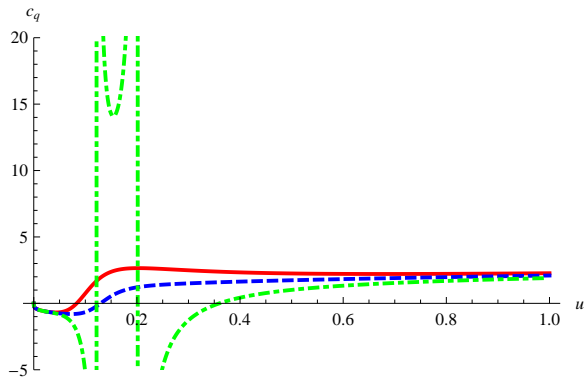


Figure 13: Normalized heat capacity c_q for $z = 2$ and $q = 0.001$ versus the event horizon of a black hole. The solid, dashed and dot-dashed curves correspond to $k = 1$, $k = 0$ and $k = -1$ respectively.

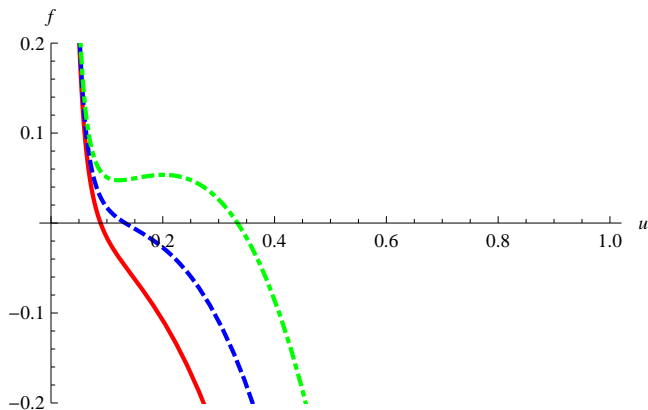


Figure 14: Normalized free energy f for $z = 2$ and $q = 0.001$ versus the event horizon of a black hole. The solid, dashed and dot-dashed curves correspond to $k = 1$, $k = 0$ and $k = -1$ respectively.

one can obtain an asymptotically Lifshitz solution in a brane-world scenario with the aid of the electric part of the Weyl tensor. The electric part of the Weyl tensor, however, reflects the global geometry of the bulk in brane-world scenarios whose effect is to enable the brane admitting an asymptotically Lifshitz geometry. We have also studied the thermodynamical behavior of such black holes. From the thermodynamical considerations we found that the dynamical exponent must be bounded from above. Together with the lower limit for the dynamical exponent z which is implied by the existence of the metric, we found that the dynamical exponent of the asymptotically Lifshitz space-time cannot possess any arbitrary values. As a result, the maximum value for z , that is, $z = \frac{11}{5}$ corresponds to the case where the entropy becomes infinite. From the maximum entropy principle, we then expected that an asymptotic Lifshitz black hole should approach the maximum value of z where the entropy is maximum. The case $z = 2$ was used to find the behavior of black holes in this case. The case $z = 2$ also has an interesting property for $k = -1$ in that the region between the two singularities in heat capacity has a negative temperature. The physical meaning of these cases are not clear yet.

6 Appendix

As is well-known, the standard 4D Einstein field equations do not admit the Lifshitz space-time even in the presence of a perfect fluid. We first note that, as we mentioned in the Introduction, the Lifshitz solution is obtained in thick brane models with an anisotropic perfect fluid. In this Appendix we show that the RS brane-world scenario can admit the Lifshitz space-time as a solution, in the presence of a perfect fluid.

The energy-momentum tensor of a perfect fluid is defined as

$$\tau_{\mu\nu} = \rho(r)u_\mu u_\nu + p(r)h_{\mu\nu}, \quad (69)$$

where $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$. Using equation (7), the tensor $\pi_{\mu\nu}$ reduces to

$$\pi_{\mu\nu} = \frac{1}{12}\rho[u_\mu u_\nu + (\rho + 2p)h_{\mu\nu}]. \quad (70)$$

Also, the electric part of the Weyl tensor is given by equation (16). The resulting field equations

are obtained using equation (6) with the result

$$\begin{aligned}
\frac{3}{l^2} + \Lambda + \rho \left(\kappa_4^2 + \frac{\kappa_5^4}{12} \rho \right) + \kappa^4 U &= 0, \\
\frac{2z+1}{l^2} + \Lambda - \kappa_4^2 p - \frac{\kappa_5^4}{12} \rho (\rho + 2p) - \frac{\kappa^4}{3} (U + 2P) &= 0, \\
\frac{z^2 + z + 1}{l^2} + \Lambda - \kappa_4^2 p - \frac{\kappa_5^4}{12} \rho (\rho + 2p) - \frac{\kappa^4}{3} (U - P) &= 0.
\end{aligned} \tag{71}$$

The solution for the above system is

$$\begin{aligned}
p = -\rho = 6 \frac{\kappa^2}{\kappa_5^2}, \quad \Lambda = -\frac{z^2 + 2z + 3}{l^2} + 6\kappa^4, \\
U = \frac{z^2 + 2z - 3}{2\kappa^4 l^2}, \quad P = \frac{z(1-z)}{\kappa^4 l^2}.
\end{aligned} \tag{72}$$

This result implies that a RS brane, in the presence of a perfect fluid, admits the Lifshitz space-time. However, the Lifshitz space-time is not a solution to the vacuum field equations. This becomes more clear if one notices that the equation related to the energy density and pressure of the perfect fluid depends on the coupling constants of the theory, which is exactly the reason why the Lifshitz space-time cannot be a vacuum solution in brane-world scenarios.

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